



Agboola, Sunday Olanrewaju

Department of Mathematical Sciences, KolaDaisi University, Ibadan

Received: March 16, 2018 Accepted: July 25, 2018

Abstract: A general arrival and Markovian service time queueing system with one server under first come first served discipline was considered, where the ij element of transition probability is given as matrix F and the system can accommodate infinite number of arrival. The steady state transition probability were obtained and compared with the result from Markovian arrival and Markovian service times queueing system (M/M/1). The formulae for waiting time distribution, probability distribution and density functions for the response time (total system time) in a G/M/1 system were obtained. Illustrative numerical examples were demonstrated to shown its usefulness in solving the real life problem. We arrived at the following values for the root of equation of Z-transform of the number of service completions that occur during an interarrival period, $\xi^j = 0.600, 0.6115, 0.6153, 0.6206, 0.6223, 0.6267, \dots$, Which is eventually converges to $\xi = 0.645705$. Therefore, the probabilities that it contains zero, one or two customers are given, respectively by 0.4, 0.12 and 0.182. The probability that arrival find the system empty is 0.5, the mean number of customers seen by an arrival is 1.0 and the mean time spent waiting in this system is 0.2.

Keywords: Transition probability matrix, Markovian service, general arrival, PASTA property

Introduction

The G/M/1 queue is a single-server queue, the service process has an exponential distribution with mean service time, $1/\mu$, i.e., $B(x) = 1 - e^{-\mu x}$, $x \geq 0$, (Medhi, 1980); while the arrival process is general with mean inter-arrival time equal to $1/\lambda$. Customers arrive individually and their inter-arrival times are independent and identically distributed. As a result, the notation G/M/1 is sometimes used to stress this independence of arrivals. We shall denote the arrival distribution by $A(t)$ and its probability density function by $a(t)$.

To represent this system by a Markov chain, it is necessary to keep track of the time that passes between arrivals, since the distribution of interarrival times does not in general possess the memoryless property of the exponential. As was the case for the M/G/1 queue, a two-component state descriptor may be used; the first to indicate the number of customers present and the second to indicate the elapsed time since the previous arrival. In this way, the G/M/1 queue can be solved using the method of supplementary variables. It is also possible to define a Markov chain embedded within the G/M/1 queue, and this is the approach that we shall follow here. The embedded time instants are precisely the instants of customer arrivals, since the elapsed interarrival time at these moments is known – it is exactly equal to zero. This allows us to form a transition probability matrix and to compute the distribution of customers as seen by an arriving customer. Unfortunately, the PASTA property no longer holds (we do not have Poisson arrivals) and so we cannot conclude that the distribution as seen by an arrival is the same as that seen by a random observer: indeed they are not the same.

Nomenclature

- $B(x)$: Service Process
- $A(t)$: Arrival Distribution
- $a(t)$: Arrival Probability Density Function
- M_k : The number of Customers Present in a G/M/1 queue just Prior to the k^{th} Arrival
- B_{k+1} : The number of Service Completions that occur between the arrival of the k^{th} customer and that of the $(k + 1)^{th}$ customer
- f_{ij} : The ij element of the transition probability matrix F
- β_i : The probability that i customers complete their service during the period k^{th} and $(k + 1)^{th}$ arrivals

- π_i : Stationary probability of an arrival finding i customers already present
- $G_B(z)$: Z-transform of the number of service completions that occur during an interarrival period
- ξ : The root of the equation $G_B(z)$
- $E[N_A]$: The mean number in this system at arrival epoch
- λ : Arrival rate
- W_q : Waiting time distribution function of the customer in the queue
- w_q : Mean time waiting in the system

Materials and Methods

We shall construct the transition probability matrix of the Markov chain embedded at arrival instants and write its solution. Let M_k be the number of customers present in a G/M/1 queue just prior to the k^{th} arrival. Let B_{k+1} be the number of service completions that occur between the arrival of the k^{th} customer and that of the $(k + 1)^{th}$ customer. It follows that

$$M_{k+1} = M_k + 1 - B_{k+1} \text{ (William, 2009).}$$

The ij element of the transition probability matrix F , namely, $f_{ij} = Prob\{M_{k+1} = j | M_k = i\}$,

is equal to the probability that $i + 1 - j$ customers are served during an arbitrary interarrival time.

This must be equal to zero when $i < j - 1$ (the second of two consecutive arrivals cannot find more than one additional customer to the number that the first finds). Also, an arrival can find any number of customers from a minimum of zero to a maximum of one more than its predecessor finds. In other words, the transition probability matrix has a lower Hessenberg structure. The probability that i customers complete their service during the period between the k^{th} and $(k + 1)^{th}$ arrivals is given by

$$\beta_i = Prob\{B_{k+1} = i\} = \int_0^\infty e^{-\mu t} \frac{(\mu t)^i}{i!} dA(t), \text{ (William, 2009)}$$

and, given the independence and identical distribution of interarrival times, the transition probability matrix is given by

$$F = \begin{pmatrix} 1 - \beta_0 & \beta_0 & 0 & 0 & \dots \\ 1 - \sum_{i=0}^1 \beta_i & \beta_1 & \beta_0 & 0 & \dots \\ 1 - \sum_{i=0}^2 \beta_i & \beta_2 & \beta_1 \beta_1 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is apparent that this matrix is irreducible and a periodic when $\beta_0 > 0$ and $\beta_0 + \beta_1 < 1$.

It turns out that the solution to this Markov chain is geometric in form. In other words, there exists a ξ ,

$$0 < \xi < 1,$$

such that

$$\pi_i = C\xi^i, \quad i \geq 0,$$

where π_i is now the stationary probability of an arrival finding i customers already present. To show that this is the case, we shall replace π_i with $C\xi^i$ in the system of equations, $\pi = \pi F$, and find the restrictions that this imposes on ξ so that $\pi_i = C\xi^i$ is indeed the solution we seek. Extracting the k^{th} equation from this system of equations, we find

$$\pi_k = \sum_{i=0}^{\infty} \pi_i f_{ik} = \sum_{i=k-1}^{\infty} \pi_i f_{ik} = \sum_{i=k-1}^{\infty} C\xi^i f_{ik} \quad \text{for } k \geq 1.$$

We do not need to be concerned with the case $k = 0$, since in considering

$\pi(F - I) = 0$, the first equation is a linear combination of all the others. Substituting $\pi_k = C\xi^k$ gives

$$C\xi^k = \sum_{i=k-1}^{\infty} C\xi^i \beta_{i+1-k} \quad \text{for } k \geq 1.$$

Now, when we divide through by $C\xi^{k-1}$, we obtain

$$\xi = \sum_{i=k-1}^{\infty} \xi^{i-k+1} \beta_{i+1-k} = \sum_{j=0}^{\infty} \xi^j \beta_j = G_B(\xi),$$

where $G_B(z)$ is the z -transform of the number of service completions that occur during an interarrival period. This then is the condition that we must impose on ξ in order for the solution to be given by $\pi_i = C\xi^i$: namely, that ξ be a root of the equation $z = G_B(z)$. One root of this system is obviously $\xi = 1$; however, we need a root that is strictly less than 1. When the steady state exists, the strict convexity of $G_B(z)$ implies that there is exactly one root of $Z = G_B(z)$ that lies strictly between zero and one: this root is the value ξ for which $C\xi^i$ is the probability of an arrival finding i customers already present. Furthermore,

$$\xi = \sum_{j=0}^{\infty} \xi^j \beta_j = \sum_{j=0}^{\infty} \xi^j \int_0^{\infty} \frac{(\mu t)^j}{j!} e^{-\mu t} dA(t) = \int_0^{\infty} e^{-(\mu - \mu\xi)t} dA(t) = F_A^*(\mu - \mu\xi) \quad \text{(Bolch, 1998)}$$

Observe that the right-hand side is the Laplace transform of the probability density function of interarrival time, evaluated at the point $\mu - \mu\xi$. The solution ξ to this functional equation may be obtained by successively iterating with

$$\xi^{(j+1)} = F_A^*(\mu - \mu\xi^{(j)}) \quad \text{----- (2.1)}$$

and taking the initial approximation, $\xi^{(0)}$, to lie strictly between zero and one. As for the constant C , it may be determined from the normalization equation. We have

$$1 = \sum_{j=0}^{\infty} C\xi^j = C \frac{1}{1 - \xi}$$

i.e., $C = (1 - \xi)$. This leads to conclude that

$$\pi_i = (1 - \xi)\xi^i.$$

It is impossible not to recognize the similarity between this formula and the formula for the number of customers in an $M/M/1$ queue, namely, $p_i = (1 - \rho)\rho^i, i \geq 0$, where $\rho = \lambda/\mu$. It follows by analogy with the $M/M/1$ queue that performance measures, such as the mean number of customers present, can be obtained by replacing ρ with ξ in the corresponding formulae for the $M/M/1$ queue.

Thus, for example, whereas $1 - \rho$ is the probability that no customers are present in an $M/M/1$ queue, $1 - \xi$ is the probability that an arrival in a $G/M/1$ queue finds it empty; the mean number in an $M/M/1$ queue is $\rho/(1 - \rho)$ and the variance is $\rho/(1 - \rho)^2$, while the mean and variance of the number of customers seen by an arrival in a $G/M/1$ queue are $\xi/(1 - \xi)$ and $\xi/(1 - \xi)^2$, respectively, and so on. It is important to remember, however, that π is the probability distribution as seen by an arrival to a $G/M/1$ queue and that this is not equal to the stationary distribution of this queue. If, indeed, the two are the same, it necessarily follows that $G = M$. Finally, results for the stationary distribution of customers in a $G/M/1$ queue—the equilibrium distribution as seen by a random observer, rather than that seen by an arriving customer – are readily available from the elements of the vector π . First, although $\pi_0 = 1 - \xi$ is that probability that an arrival finds the system empty, the stationary probability of the system being empty is actually $p_0 = 1 - \rho$.

Furthermore, the stationary probability of a $G/M/1$ queue having $k > 0$ customers is given by

$$p_k = \rho(1 - \xi)\xi^{k-1} = \rho\pi_{k-1} \quad \text{for } k > 1 \quad \text{(Kleinrock, 1975).}$$

Thus once the variable ξ has been computed from Equation (2.1), the stationary distribution is quickly recovered.

Waiting time distributions in a $G/M/1$ queue

The distribution of customers at arrival epochs is generally not that which is sought (the distribution of customers as seen by a random observer being the more usual), it is exactly that which is needed to compute the distribution of the time spent waiting in a $G/M/1$ queue before beginning service. When the scheduling policy is first come, first served, an arriving customer must wait until all the customers found on arrival are served before this arriving customer can begin its service. If there are n customers already present, then an arriving customer must wait through n services, all independent and exponentially distributed with mean service time $1/\mu$. An arriving customer that with probability $\pi_n = (1 - \xi)\xi^n$ finds $n > 0$ customers already present, experiences a waiting time that has an Erlang- n distribution. There is also a finite probability $\pi_0 = 1 - \xi$ that the arriving customer does not have to wait at all. The probability distribution function of the random variable T_q that represents the time spent waiting for service will therefore have a jump equal to $1 - \xi$ at the point $t = 0$.

For $t > 0$, we may write

$$W_q(t) = \text{Prob}\{T_q \leq t\} = \sum_{n=1}^{\infty} \int_0^t \frac{\mu(\mu x)^{n-1}}{(n-1)!} e^{-\mu x} dx (1 - \xi)\xi^n + (1 - \xi)$$

$$= \xi(1 - \xi) \int_0^t \mu e^{-\mu x(1-\xi)} dx + (1 - \xi)$$

$$= 1 - \xi e^{-\mu(1-\xi)t}, \quad t > 0 \quad \text{(Lucantoni, 1993).}$$

This formula also gives the correct result $1 - \xi$ when $t = 0$ and so

$$W_q(t) = 1 - \xi e^{-\mu(1-\xi)t}, \quad t \geq 0, \quad \text{----- (3.1)}$$

The mean waiting time is

$$W_q = E[T_q] = \frac{\xi}{\mu(1-\xi)}. \quad \text{----- (3.2)}$$

These results could be obtained directly from the corresponding results for the $M/M/1$ queue by simply replacing ρ with ξ .

The mean waiting time in $M/M/1$ queue is given by

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}.$$

Since $\rho = \frac{\lambda}{\mu}$, replacing λ with $\mu\xi$ gives Equation (3.2).

Numerical example

To show that $\xi = \rho$ when the arrival process in a G/M/1 queue is Poisson, i.e., when $G = M$.

Solution:

We need to solve the functional equation

$$\xi = F_A^*(\mu - \mu\xi) \quad \text{when} \quad a(t) = \lambda e^{-\lambda t} dt = -\frac{\lambda}{s+\lambda} e^{-(s+\lambda)t} \Big|_0^\infty = \frac{\lambda}{s+\lambda},$$

So that the function equation becomes

$$\xi = \frac{\lambda}{\mu(1 - \xi) + \lambda}$$

Or

$$(\xi - 1)(\mu\xi - \lambda) = 0,$$

With the two solution $\xi = 1$ and $\xi = \frac{\lambda}{\mu} = \rho$.

Only the latter solution satisfies the requirement that $0 < \xi < 1$.

That is $\xi = \rho$ when the arrival process in a G/M/1 queue is Poisson.

In the case, the interarrival time is constant and equal to $\frac{1}{\lambda}$.

The probability distribution function has the value 0 for $t < \frac{1}{\lambda}$ and has the value 1 for $t \geq \frac{1}{\lambda}$. The density function is a Dirac impulse at the point $t = \frac{1}{\lambda}$ and its Laplace transform is known to be

$$F_A^*(s) = e^{-\frac{s}{\lambda}}.$$

Solution

The functional equation we need to solve for ξ is

$$\xi = e^{-(\mu - \mu\xi)/\lambda} = e^{-(1 - \xi)/\rho}$$

To proceed any further it is necessary to give a numeric value to ρ and solve using an iterative procedure. Assuming $\rho = \frac{4}{5}$ and iterative process begin with

$$\xi^{(0)} = 0.6$$

With successive iterations of

$$\xi^{(j+1)} = \exp\left(\frac{\xi^{(j)} - 1}{0.8}\right)$$

Gives

$$\text{when } j = 0, \xi^{(1)} = \exp(-0.500) = 0.6065$$

$$\text{when } j = 1, \xi^{(2)} = \exp(-0.4918) = 0.6115$$

$$\text{When } j = 2, \xi^{(3)} = \exp(-0.4856) = 0.6153$$

$$\text{When } j = 3, \xi^{(4)} = \exp(-0.4771) = 0.6206$$

$$\text{When } j = 4, \xi^{(5)} = \exp(-0.4743) = 0.6223$$

$$\text{When } j = 5, \xi^{(6)} = \exp(-0.4721) = 0.6267$$

$$\xi^j = 0.600, 0.6115, 0.6153, 0.6206, 0.6223, 0.6267, \dots,$$

Which is eventually converges to $\xi = 0.645705$ (Agboola, 2016).

The mean number in this system at arrival epoch is given by

$$E[N_A] = \frac{\xi}{1 - \xi} = 1.82257.$$

The probability that an arrival to this system finds it empty

$$p_0 = 1 - \xi = 0.35429$$

Given a G/M/1 queue in which the service time is exponential

Solution

The Laplace transform of this distribution is given by

$$F_A^*(s) = \left(\frac{\lambda_1}{s+\lambda_1}\right) \left(\frac{\lambda_2}{s+\lambda_2}\right).$$

Its expectation is $\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, this gives $\rho = \frac{6/5}{2} = \frac{3}{5}$.

Substituting the value of λ_1 and λ_2 , we obtain

$$F_A^*(s) = \frac{6}{(s+2)(s+4)}$$

So that

$$F_A^*(\mu - \mu\xi) = \frac{6}{(2 - 2\xi + 2)(2 - 2\xi + 4)} = \frac{3}{(\xi - 2)(\xi - 3)},$$

And the fixed point equation $\xi = F_A^*(\mu - \mu\xi)$ now becomes

$$2\xi^3 - 9\xi^2 + 10\xi - 3 = 0$$

This imply

$$(\xi - 1)(2\xi^2 - 7\xi + 3) = 0$$

Which has the three roots $\frac{1}{2}$, 1 and 3.

Since we need the root that is strictly less than 1, which is only satisfy by the root $\xi = \frac{1}{2} = 0.5000$.

Therefore, the probabilities that it contains zero, one, or two customers are given, respectively, by

$$p_0 = 1 - \rho = 1 - \frac{3}{5} = \frac{2}{5},$$

$$p_1 = \rho(1 - \xi)\xi^0 = \frac{3}{5} \left(1 - \frac{1}{2}\right) 0.6 = 0.1200,$$

$$p_2 = \frac{3}{5} \rho \left(1 - \frac{1}{2}\right) 0.6065 = 0.182;$$

The probability that an arrival finds the system empty is $(1 - \xi) = 0.500$;

The mean number of customers seen by an arrival is $\frac{\xi}{(1 - \xi)} = 0.500$ and so on

From Numerical example 4.3, we can write the distribution function of the G/M/1 queue as

$$W_q(t) = 1 - \left(\frac{1}{2}\right) e^{-(\frac{5}{2})t} = \frac{1}{2} e^{-(\frac{5}{2})t}; \quad t \geq 0,$$

which has the value $\frac{1}{2}$ when $t = 0$.

The mean time spent waiting in this system is

$$W_q = \frac{\xi}{\mu(1 - \xi)} = \frac{0.5}{5(1 - 0.5)} = \frac{1}{5}$$

Conclusion

In this research, the steady state transition probability matrix were obtained for G/M/1 queue and compared with the result from Markovian arrival and Markovian service times queueing system (M/M/1). The formulae for waiting time distribution, probability distribution and density functions for the response time (total system time) in a G/M/1 system were obtained. Illustrative numerical examples were demonstrated to shown its usefulness in solving the real life problem and we arrived at $\xi^j = 0.600, 0.6115, 0.6153, 0.6206, 0.6223, 0.6267, \dots$, which is eventually converges to $\xi = 0.645705$.

Therefore, the probabilities that it contains zero, one or two customers are given, respectively by 0.4, 0.12 and 0.182. The probability that arrival find the system empty s obtained 0.5, the mean number of customers seen by an arrival is 1.0 and the mean time spent waiting in this system is 0. 2.

References

Agboola SO 2016. Repairman Problem with Multiple Batch Deterministic Repairs. Unpublished Ph.D. Mathematics (Statistics Option) Thesis submitted to Department of Mathematics, Obafemi Awolowo University, Ile-Ife.

Agboola SO 2007. On the Analysis of M/G/1 Queues. Unpublished M.Sc. (Mathematics) Thesis submitted to Department of Mathematics, University of Ibadan.

Agboola SO 2011. The Analysis of Markov Inter-arrival Queues Model with K – Server under Various Service Points. Unpublished M.Sc. (Statistics) Thesis submitted to Department of Mathematics, Obafemi Awolowo University, Ile-Ife.

Bolch G, Greiner S & Trivedi KS 1998. Queueing Networks and Markov Chains. Wiley Interscience, New York.

- Jain M 2013. Transient analysis of machining system with service interruption, mixed standbys and priority. *Int. J. Maths. Operations Res.*, 5(5): 604-625.
- Jain M & Singh M 2013. Bi-level control of degraded machining system with warm standby setup and vacation. *Applied Mathematical Modelling*, 28: 1015 - 1026.
- Kleinrock L 1975. Queueing systems, New York, pp. 60 - 158.
- Law AM & Kelton WD 2000. Simulation Modeling and Analysis. Third Edition, McGraw-Hill, New York.
- Lucantoni DM 1993. The BMAP/G/1 Queue: Models and Techniques for performance evaluation of computer and communications systems. Springer.
- Medhi J 1980. Stochastic Application of Queuing Theory. Guahandi University, Guwa-hata, Indian, pp. 23 - 120.
- Meini B 1997. New convergence results on functional techniques for the numerical solution of $M/G/1$ Type Markov Chains. *Numer. Math.*, 78: 39-58.
- Ross SM 1997. Simulation. Second edition, Academic Press, New York.
- Saad Y 2003. Iterative Methods for Sparse Linear Systems. Second edition, SIAM Publications, Philadelphia.
- William Stewart 2009. Probability, Markov Chains, Queueing and Simulation, Princeton University Press, Princeton and Oxford.